Econ 101A – Fianl Review

Clotaire Boyer - UC Berkeley

Setup. A firm produces a single good y with Cobb–Douglas technology $y = K^\alpha L^\beta$, $\alpha, \beta > 0$, with input prices r (capital) and w (labor). For all market analysis (Q2–Q6) we will work with the reduced-form cost function

$$c(y) = cy + \frac{\varphi}{2} y^2, \qquad c \geq 0,\ \varphi \geq 0,$$

and inverse market demand $p(Y) = a - bY$ with $a > c > 0,\ b > 0$. Denote $A \equiv a - c$.

Q1. Production theory warm-up. ⋆⋆

- **(a)** Classify the returns to scale of $y = K^\alpha L^\beta$ in terms of $\alpha + \beta$. In one sentence, state what the classification implies for the shape of long-run average cost.
- **(b)** Compute the marginal rate of technical substitution $\mathrm{MRTS}_{L,K} = \mathrm{MP}_L/\mathrm{MP}_K$. Show that for Cobb–Douglas it depends only on the ratio K/L, not on the level of output. Interpret economically.
- **(c)** Set up the cost-minimization Lagrangian $\min wL + rK$ s.t. $K^\alpha L^\beta = y$. State the FOCs and the tangency condition. Interpret the multiplier λ^*.
- **(d)** Specialize to $\alpha = \beta = \frac{1}{2}$ (CRS). Derive the cost function $c(y; w, r)$ in closed form, and report the marginal cost. Why is the cost function linear in y here?

> **How to: the Lagrangian, multipliers, and the envelope theorem**
>
> For $\max f(x)$ s.t. $g(x) = b$, write $\mathcal{L} = f(x) - \lambda(g(x) - b)$ and solve $\nabla f = \lambda \nabla g$, $g(x) = b$. The multiplier λ^* is the *shadow price*: $\partial V^*/\partial b = \lambda^*$, where $V^*(b)$ is the value function. This is the *envelope theorem*: when you change a parameter of the constraint by db, only the direct effect through the constraint matters at the optimum (the indirect effects through the optimal x^* vanish because of the FOC). In cost minimization, $\lambda^* = \partial c^*/\partial y = \mathrm{MC}(y)$ – the multiplier *is* the marginal cost.

Q2. Market equilibrium and tax incidence. ⋆⋆⋆

There are N identical firms with the reduced-form cost above ($\varphi > 0$); aggregate supply is $Y_S(p) = N(p - c)/\varphi$. The government imposes a per-unit tax $t \geq 0$ on *output*, paid by firms.

- **(a)** Solve for equilibrium consumer price $p^*(t)$ and quantity $Y^*(t)$. Compute $\frac{\partial p^*}{\partial t}$ and the price the producer receives. State the share of the tax borne by each side.
- **(b)** Show that $\frac{\partial p^*}{\partial t} = \dfrac{\varepsilon_S}{\varepsilon_S - \varepsilon_D}$, where $\varepsilon_S, \varepsilon_D$ are the supply and demand elasticities at equilibrium. State the limiting cases $\varphi \to 0$ (perfectly elastic supply) and $\varphi \to \infty$ (perfectly inelastic supply), and give the standard one-line lesson about which side bears the burden.
- **(c)** Compute the deadweight loss $\mathrm{DWL}(t)$ as a closed-form expression in (t, b, φ, N). Show that $\partial \mathrm{DWL}/\partial t$ vanishes at $t = 0$, and explain in one sentence why.

> **How to: the Implicit Function Theorem (IFT) for comparative statics**
>
> Let an FOC implicitly define $x^*(\theta)$ via $F(x^*, \theta) = 0$. Then
>
> $$\frac{\partial x^*}{\partial \theta} = -\frac{F_\theta(x^*, \theta)}{F_x(x^*, \theta)}.$$
>
> For a maximization problem, $F = \partial \pi/\partial x$, so the denominator is the second-order condition (negative). The sign of the comparative static is then the same as the sign of the numerator F_θ. Use this engine for: input-demand comparative statics, monotone Marshallian demand, and – crucially – state-contingent expected-utility problems (Q5).

Q3. An externality and the corrective tax. ⋆⋆

Suppose now that each unit of Y produces one unit of pollution; pollution causes constant marginal damage $d > 0$ to consumers, but is not priced.

- **(a)** Write the social planner's problem (consumer surplus + producer profit − aggregate damage). Derive Y^{SO} and contrast its FOC with the laissez-faire FOC. Identify the wedge.
- **(b)** Show that a Pigouvian tax $t^{\mathrm{Pig}} = d$ exactly decentralizes Y^{SO} in the competitive economy. In one sentence, explain in plain words why an unfettered market overproduces here.
- **(c)** *Coase one-liner.* Under what assumption would a private negotiation between firms and consumers reach Y^{SO} *without* any tax? Name the assumption that breaks down in real-world pollution problems.

Q4. Strategic interaction: monopoly, Cournot, Bertrand, Stackelberg. ⋆⋆⋆⋆

For this question only, set $\varphi = 0$ so each firm has constant marginal cost c. (Two firms in (b)–(d).)

- **(a)** *Monopoly.* A single firm chooses Y to maximize profit. Derive Y^M, p^M, π^M. Compute the Lerner index $(p - MC)/p$ at the optimum.
- **(b)** *Cournot duopoly.* Two firms choose quantities simultaneously. Derive each firm's reaction function and the symmetric Nash equilibrium (y^C, Y^C, p^C, π^C).
- **(c)** *Bertrand duopoly.* Same two firms now choose prices simultaneously. State the unique Nash equilibrium price and profit. In two lines, explain why this dramatic discontinuity vs. Cournot occurs ("Bertrand paradox").
- **(d)** *Stackelberg.* Now firm 1 chooses y_1 first; firm 2 observes y_1 and chooses y_2. Solve by backward induction: first the follower's reaction $y_2^*(y_1)$, then the leader's optimal y_1^S. Compute Y^S, p^S, π_1^S, π_2^S.
- **(e)** Rank $Y^M < Y^C < Y^S < Y^B = Y^{PC}$ and the corresponding prices. State the key economic intuition for the Stackelberg first-mover advantage in two lines.

How to: backward induction in dynamic games

In a sequential game, solve from the *last* stage backwards. (i) Take the leader's action as fixed and solve the follower's problem – this gives a reaction function $y_2^*(y_1)$. (ii) Substitute this reaction into the leader's payoff so the leader's problem is now a one-player optimization in y_1 alone. (iii) Solve the leader's FOC. The resulting profile is the (unique) subgame-perfect equilibrium. The leader exploits commitment: her FOC *internalizes* the follower's reaction, which the simultaneous-move (Cournot) player cannot.

Q5. Risk aversion and insurance demand. ⋆⋆⋆

A consumer with wealth $w > 0$ and Bernoulli utility u ($u' > 0$, $u'' < 0$) faces a loss of $L > 0$ with probability $p \in (0, 1)$. An insurer offers coverage $\alpha \in [0, L]$ at premium $q\alpha$, where $q > 0$.

- **(a)** Set up the expected-utility maximization problem. Assuming an interior solution, write the FOC.
- **(b)** *Fair insurance.* If $q = p$, show that $\alpha^* = L$ (full insurance). Provide the concavity argument in one line.
- **(c)** *Loaded premium.* Suppose $q > p$, so the optimum is interior. Apply the IFT to the FOC $F(\alpha, w) = 0$ and derive an expression for $\partial\alpha^*/\partial w$ in terms of u', u'' at $w_1 = w - q\alpha^*$ and $w_2 = w - q\alpha^* - L + \alpha^*$.
- **(d)** Use the FOC to substitute one u' in your expression. Show that the sign of $\partial\alpha^*/\partial w$ depends entirely on the sign of $r_A(w_1) - r_A(w_2)$, where $r_A(w) \equiv -u''(w)/u'(w)$ is Arrow–Pratt absolute risk aversion.
- **(e)** Conclude: under *DARA* (decreasing absolute risk aversion – Arrow's hypothesis, supported empirically), is $\partial\alpha^*/\partial w$ positive or negative? Give the policy implication for compensation programs that target heterogeneous-wealth victims.

How to: state-contingent expected utility & IFT comparative statics

(i) Write wealth in each state as a function of the choice; (ii) maximize $\mathbb{E}[u(\widetilde{w})]$; (iii) the FOC equates probability-weighted marginal utilities at *different* wealth levels. For comparative statics, $\partial\alpha^*/\partial\theta = -F_\theta/F_\alpha$. The denominator is the SOC (negative). The numerator typically simplifies, after *re-using the FOC* to eliminate one of the marginal utilities, into a comparison of $r_A(\cdot)$ at two wealth levels. *Plugging the FOC back into the comparative-static expression* is the workhorse trick across consumer theory (Slutsky), producer theory (Le Chatelier), and asset pricing.

Stretch (intertemporal expected utility). The same consumer now lives two periods and discounts future utility at rate δ. With probability p, period-1 income falls by L; otherwise it is unchanged. Assets earn safe gross return $1 + r$. Write the period-0 problem $\max_s \; u(c_0) + \frac{1}{1+\delta}\mathbb{E}[u(c_1)]$ subject to the period-budget constraints. Show the consumption Euler equation $u'(c_0) = \frac{1+r}{1+\delta}\,\mathbb{E}[u'(\widetilde{c}_1)]$. (No need to solve fully; the message is that the same FOC machinery operates, with discounting in place of the loaded premium.)

Q6. Walrasian equilibrium and the First Welfare Theorem. ⋆⋆⋆

Two consumers $i \in \{A, B\}$ have endowments $\omega^i = (\omega_1^i, \omega_2^i)$ and Cobb–Douglas preferences $u^i(x_1, x_2) = x_1^{a_i} x_2^{1-a_i}$.

- **(a)** Write the Lagrangian for consumer i. Derive the demand for good 1 as a function of (p_1, p_2) and the endowment income $m^i = p_1\omega_1^i + p_2\omega_2^i$.
- **(b)** Use market clearing in good 1 (Walras' law gives the other) to obtain the equilibrium price ratio p_2^*/p_1^*.
- **(c)** *Verify the First Welfare Theorem.* At the Walrasian equilibrium, compute MRS^A and MRS^B and check that they are equal. Why is this enough to conclude Pareto efficiency?
- **(d)** State the FWT in one line. In one further line, explain why it *fails* once production of one of the goods generates an externality on a third party.