# Econ 101A
# Section 11

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# 1 Production Functions

## 1.1 Definition and Description

- A **production function** $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ maps *inputs* (factors of production) to *outputs* (goods and services).
- Usually, we consider the two input case where some good $y = f(K, L)$ is produced using capital $K$ and labor $L$.
- Common assumptions about production functions:
  - $f(\mathbf{0}) = 0$ ("no free lunch" - if you have no inputs, you will produce nothing)
  - $\frac{\partial f}{\partial z_i} > 0$ for all $i$ (positive marginal productivity for all inputs $z_i$)
  - $\frac{\partial^2 f}{\partial z_i^2} < 0$ for all $i$ (decreasing marginal productivity for all inputs $z_i$)
  - (in two-input case) $\frac{d^2 K}{dL^2} > 0$ (convex isoquants, which implies a convex production function)[1]

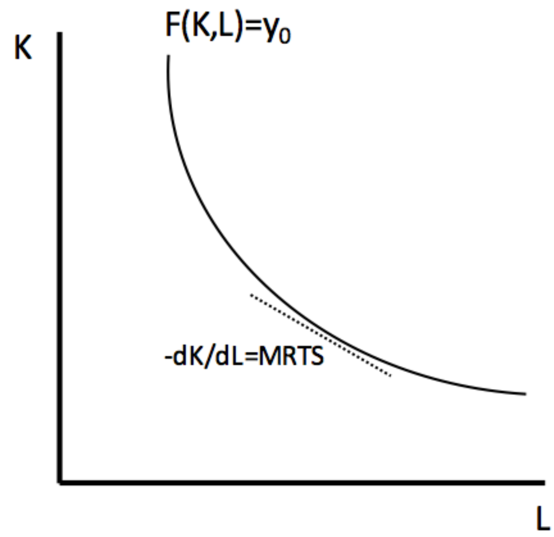
## 1.2 Isoquants, two variable case (compare to indifference curves!)

- An **isoquant** is a curve representing different combinations of inputs that yield the same amount of output, or the line represented by the equation $\boxed{f(K, L) = y_0}$ for the output level $y_0$.
- The **marginal rate of technical substitution (MRTS)** is the rate at which we substitute out capital for an additional unit of labor to remain at the same level of production.
- MRTS is also the absolute value of the slope of the isoquant. Assuming positive marginal productivity, and applying the Implicit Function Theorem, we get

$$MRTS = -\frac{dK}{dL} = \frac{\partial f / \partial L}{\partial f / \partial K}$$

- Below is an example isoquant graph (assuming all assumptions given in Section 1.1). often graph isoquants with capital $K$ as the vertical axis and labor $L$ as the horizontal axis. Notice the following, which are similar to indifference curves:
  - As we move northeast (on the graph), we reach isoquants that indicate higher levels of production
  - Our isoquants are convex, indicating that a mixture of capital and labor will allow us to produce more than using *only* capital or *only* labor.

[1] Note that a sufficient condition for this to hold is that $\frac{\partial^2 f}{\partial K \partial L} > 0$ - this says that the marginal productivity of labor increases with more capital and vice versa.




- Note: *unlike* with utility, you *cannot* do monotonic transformations of a production function and expect the same result.
  - Recall that in utility maximization problems without uncertainty, the utility function $U(x, y) = (x + y)^{1/2}$ would lead to the same choices as $U(x, y) = (x + y)$ or $U(x, y) = (x + y)^2$. These functions represent the same preferences (excluding risk preferences), since each function is a monotonic transformation of the others.
  - In utility maximization problems, only the *nominal* level of $U$ would be affected, and this had no interpretable meaning - we only care about the *order* when thinking about preferences.
  - In production problems, however, $Y$ has real meaning: it is the quantity produced.
  - Thus, $f(L, K) = (L+K)^{1/2}$ differs meaningfully from $f(L, K) = (L+K)$ or $f(L, K) = (L+K)^2$. Each will lead to very different production decisions (notice they have different returns to scale).

### 1.3 Returns to Scale

- $f$ has **constant returns to scale** if for all $\mathbf{z}$ and for all $t > 1$, $f(t\mathbf{z}) = tf(\mathbf{z})$
- $f$ has **decreasing returns to scale** if for all $\mathbf{z}$ and for all $t > 1$, $f(t\mathbf{z}) < tf(\mathbf{z})$
- $f$ has **increasing returns to scale** if for all $\mathbf{z}$ and for all $t > 1$, $f(t\mathbf{z}) > tf(\mathbf{z})$

Given a production function $f(k, l)$, how do we determine whether it has increasing, decreasing, or constant returns to scale?

1. Obtain $f(tK, tL)$ by plugging in $tK$ for every $K$ and $tL$ for every $L$ that shows up in $f(K, L)$.
2. Rearrange terms so that you can easily compare your answer to $tf(K, L)$. When comparing, remember that $t > 1$.

## 2 Exercises

### 2.1 Isoquants

Draw one isoquant for each of the following production functions:

- $y = K^{\alpha}L^{1-\alpha}$, where $0 < \alpha < 1$
- $y = K + L$

- $y = \min\{K, L\}$

In a new graph, draw the isoquants of two production processes that exhibits (a) constant returns to scale, and (b) increasing returns to scale.

## 2.2 Returns to Scale

Determine whether the following production processes exhibit decreasing, constant, or increasing returns to scale.

(a) Special Case Perfect Substitutes: $y = (L + K)^{1/2}$

(b) General Cobb-Douglas: $f(L, K) = L^{\alpha} K^{\beta}$

## 2.3 Returns to Scale (Agriculture)

Let $f(L, K, F) = AL^{\alpha} K^{\beta} F^{\gamma}$ where $F$ is farmland.

1. Show that $f(L, K, F)$ exhibits constant returns to scale if $\alpha + \beta + \gamma = 1$. Interpret this result.
2. Consider a small farm with the production function written above, with $\alpha+\beta+\gamma = 1$. In the short-run, however, the farm owner owns a fixed amount of land $F = \bar{F}$. Thus, his short-run production function is $Y_{SR} = AL^{\alpha} K^{\beta} \bar{F}^{\gamma}$. Does this production function have increasing, decreasing, or constant returns to scale in $L$ and $K$?