

Econ 101A

Section 12

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The objective of a (neoclassical) firm is to maximize profit (revenue minus cost). This section starts with two procedures that can be used separately to achieve this goal: Two-step cost minimization and one-step profit maximization.

1 Two-Step Cost Minimization

The **first step** is to solve a cost minimization problem (the consumer theory analogy is expenditure minimization). That is, for any given output level, the firm chooses the optimal production allocation to minimize costs, which is steps 1 to 5 below. Building on cost minimization, the **second step** is to choose the optimal output level to maximize profit, which is steps 6 to 9 below.

1. Set up the cost minimization problem

$$\min_{K,L} rK + wL \quad s.t. \quad f(K,L) \geq y$$

2. Write out the Lagrangian

$$\mathcal{L} = rK + wL - \lambda [f(K,L) - y]$$

3. F.O.C.:

$$\begin{aligned} r - \lambda f'_K &= 0 \\ w - \lambda f'_L &= 0 \\ -f(K,L) + y &= 0 \end{aligned}$$

4. Solve...

5. Check S.O.C.: technically the determinant of bordered Hessian must be negative (but we would give you a linear form to check in this class so no worries on that)

6. Now set up the profit maximization problem

$$\max_y p \cdot y - [rK^*(y, r, w) + wL^*(y, r, w)]$$

(a) Alternatively, using cost function notation

$$\max_y p \cdot y - c(y, r, w)$$

7. F.O.C.:

$$p - c'_y(y^*, r, w) = 0$$

or

$$p - r \frac{\partial K^*}{\partial y} - w \frac{\partial L^*}{\partial y} = 0$$

(Note: The first step reduces many cost functions into one cost function $c_i(y, r, w)$, taking into account all the complementarity/substitutability between inputs. It therefore allows for an output decision in which the firm sets $\frac{\partial c}{\partial y}(y^*, r, w) = p$, à la Econ 1.)

8. Solve...

9. Check S.O.C.: second derivative of profit function must be negative. Since revenue function is linear, this is the same as saying that the second derivative of the cost function must be positive. That is,

$$-c''_{yy}(y^*, r, w) < 0$$

$$c''_{yy}(y^*, r, w) > 0$$

This S.O.C. implies that (optimized) marginal cost is increasing in output level, suggesting decreasing returns to scale. Note that this S.O.C. is crucial. If $c''_{yy}(y^*, r, w) < 0$, then y^* is profit-minimizing output instead of profit-maximizing output.

Understanding the S.O.C.: Take the example from class where we consider the general Cobb-Douglas production function $y = f(L, K) = AK^\alpha L^\beta$. We have solved the cost function, assuming no fixed costs:

$$c(y, r, w) = By^{\frac{1}{\alpha+\beta}} w^{\frac{\alpha}{\alpha+\beta}} r^{\frac{\beta}{\alpha+\beta}}$$

where $B = (\frac{1}{A})^{\frac{1}{\alpha+\beta}} ((\frac{\alpha}{\beta})^{-\frac{\alpha}{\alpha+\beta}} + (\frac{\beta}{\alpha})^{-\frac{\beta}{\alpha+\beta}}) > 0$. We can compute $c''(y, r, w)$:

$$c''(y, r, w) = B \left(\frac{1}{\alpha+\beta} \right) \left(\frac{1}{\alpha+\beta} - 1 \right) \times y^{\frac{1}{\alpha+\beta}-2} w^{\frac{\alpha}{\alpha+\beta}} r^{\frac{\beta}{\alpha+\beta}}$$

- If $\alpha + \beta < 1$ (DRS), then $c''(y, r, w)$ is positive, S.O.C is negative, y^* is a maximum.
- If $\alpha + \beta > 1$ (IRS), then $c''(y, r, w)$ is negative, S.O.C is positive, y^* is a minimum. As you see in class, the profit-maximization solution in $y^* \rightarrow \infty$
- If $\alpha + \beta = 1$ (CRS), then $c''(y, r, w) = 0$, S.O.C. inconclusive. But when $\alpha + \beta = 1$, you have $MC = \text{constant}$. Hence, we only need to compare p and MC . In fact,
 - If $p > MC$, then $y^* \rightarrow \infty$.
 - If $p = MC$, then $y^* \in [0, \infty)$.
 - If $p < MC$, then $y^* = 0$.

2 Cost Curves

We use the Two-Step Cost Minimization procedure to determine how much output a firm will produce as a function of input prices (w, l) and the output price p . Here, we will use graphical analysis of marginal and average cost curves to do the same thing. In some cases, when we want to find the supply function but don't particularly care about the choices of inputs, we will start with a cost function $c(y)$ rather than having to solve the first step of the cost minimization problem. This is the same as assuming that prices w and l are constant, so we can just plug their values into $c(y, r, w)$ and end up with $c(y)$.

2.1 Total Cost

- The cost function is: $TC = c(y, r, w) = wL^*(y, r, w) + rK^*(y, r, w)$
- The cost function is the value function of the cost minimization problem. It is the optimized objective. You obtain it by plugging in L^* and K^* into your objective function, $wL + rK$ of the cost minimization problem.
- Similar concepts: expenditure function (and indirect utility function) in consumer theory.

2.2 Marginal Cost

- From the cost function you can obtain the marginal cost: $MC = \partial c(y, r, w) / \partial y$ or $c'_y(y)$.
- This is the additional cost incurred to increase production by one unit, or the cost of producing one more unit when you are currently producing y .
- Why is this curve important? Recall that the f.o.c. of the second step of the Two-Step Cost Minimization problem gives us $p - c'_y(y^*) = 0$, which means $p = MC$ at the profit maximizing point *as long as the s.o.c. holds*.
- Intuition: Suppose the firm produces $y' \neq y^*$
 - If $MC(y') > p$ at a y' , then profit can be increased by decreasing y .
 - If $MC(y') < p$ at a y' , then profit can be increased by increasing y .
 - If $MC(y') = p$ at a y' , then you are at a critical point, either a profit *minimum* or a profit *maximum*. That is why it is essential to check the s.o.c.: $c''_{yy}(y^*, r, w) > 0!$

2.3 Average Cost

- From the cost function you can obtain the average cost: $AC = c(y, r, w) / y$ or $\frac{c(y)}{y}$
- Why do we also need this curve? A firm will only produce at a point where $AC < p$ because that means it is either making positive total profit or breaking even. If the firm loses money when producing y , it would rather not produce at all.
- Note that $AC < p$ is a necessary and sufficient condition for positive profit. If $AC \geq p$, then multiplying by y gives $TC = AC \cdot y \geq py = TR$ (total revenue), which is the condition for non-positive profit.
- If there are no fixed costs, then:
 - The AC and MC curves begin at the same point.

- If MC is always increasing, then AC always lies below it.
- If MC is always decreasing, then AC always lies above it.
- If MC decreases first then increases, then 1) AC also decreases first then increases 2) AC is above MC first, then below MC . Moreover, MC crosses AC at AC 's lowest point.

2.4 Supply Function

The supply function is the portion of the marginal cost curve that lies *above* the average cost curve.

- Why? Because we know the firm wants to produce enough y so that $MC = p$, and we also know that the firm will only produce anything when $AC < p$.
- So, the MC curve when it lies above AC curve represents the range of price-output combinations where $AC < p$ and $p = MC$.

3 One-step Profit Maximization

- Another way to solve the firm's problem is by **one-step profit maximization**, where we write both revenue and cost in terms of the production inputs and then choose the optimal quantity of these inputs.

$$\max_{L,K} pf(L, K) - wL - rK$$

Notice that this is an unconstrained maximization problem, and that it is multivariate (the dimension depends on how many inputs you have in the production function).

- The one-step profit-maximization problem yields optimal input decisions $L^*(p, r, w)$ and $K^*(p, r, w)$ and output decisions $y^*(p, r, w) = f(L^*, K^*)$.
- Notice that the L^* and K^* solutions you get here are *not* functions of Y as they are in the Two-Step Cost Minimization problem, instead they are functions of p . That's because in the Two-Step Cost Minimization problem we found the optimal input quantities *conditional* on reaching a certain level of output (then in the second step we found the optimal level of output). In this case, L^* and K^* are chosen to maximize profit directly, in other words, assuming that you are choosing the optimal level of output for a given price p .
- Also, the cost function you would get just by plugging in L^* and K^* is not exactly the same thing because you have price instead of output: $c^*(p, r, w) = wL^*(p, r, w) + rK^*(p, r, w)$.

4 Exercises

4.1 Fun with cost minimization

For each of the following functions, derive the total cost $c(w, r, y)$, average cost $\frac{c(w,r,y)}{y}$, and marginal costs $\frac{\partial c(w,r,y)}{\partial y}$.

1. $f(L, K) = 2L^{\frac{1}{4}}K^{\frac{1}{4}}$
2. $f(L, K) = [\min(L, K)]^{\frac{1}{3}}$

4.2 Problem 2, MT2 Fall 2004

In this exercise, we consider a firm producing product y using two inputs, labor L and capital K . The production function is $y = f(L, K) = (L + K)^\alpha$. Assume that the wage of a worker is w and the cost of capital is r . Assume $L \geq 0$, $K \geq 0$, and $\alpha > 0$.

1. Draw a picture of the isoquants. What is the unusual feature of this production function?
2. For which values of α does the function exhibit decreasing returns to scale (that is, $f(tL) < tf(L)$ for all $t > 1$ and all $L \geq 0$)?
3. Consider now the first step of the cost minimization problem. The firm solves

$$\min wL + rK \quad \text{s.t. } f(L, K) \geq y$$

for $y > 0$. What are the solutions for $L^*(w, r, y|\alpha)$ and $K^*(w, r, y|\alpha)$? (This notation stresses that the solution depends also on the parameter α . Hint: You are better off not using Lagrangeans. The pictures you drew in point 1 may be helpful)

4. Write down the implied cost function $c(w, y|\alpha)$.
5. Derive an expression for the average cost $c(w, y|\bar{L}, \alpha)/y$ and the marginal cost $c'_y(w, y|\bar{L}, \alpha)$ for $y > 0$ and $w < r$. Graph the average cost and marginal cost for $\alpha = .5$, $w = 1$, and $r = 2$. Graph the supply function for the same values of the parameters. [remember, y is on the horizontal axis].
6. Now that we graphically solved for the supply function, we also derive it formally for all $\alpha > 0$. Consider the second step of cost minimization

$$\max_y py - c(w, y|\bar{L}, \alpha).$$

Write down the first order condition and the second order conditions. Solve for $y^*(w, p|\bar{L}, \alpha)$. (here do *not* assume $w < r$) For what values of α is the second order condition satisfied?

7. From now on, assume $\alpha < 1$. Take the solution for $y^*(w, p|\bar{L}, \alpha)$ in point 6 and consider what happens to y^* as the wage w increases. Obtain the sign of $\partial y^*/\partial w$ for the cases $w < r$ and $w > r$. Provide intuition on this result.
8. Consider now what happens to the supply function as price of output p increase. Obtain the sign of $\partial y^*/\partial p$ and provide intuition on the result.
9. Does the company make, negative, or positive profits for $p > 0$? Provide an argument for your answer.

4.3 Production with Fixed Costs

Suppose a firm has a cost function $c(y) = H + y^2$ if $y > 0$ and $c(0) = 0$.

1. Derive and draw the MC and AC curves.
2. Solve for y at the point where the two curves intersect, and show that this point is also the minimum of the AC curve.

3. Draw the supply function for the firm with cost function $c(y) = H + y^2$ if $y > 0$ and $c(0) = 0$, and write an expression representing it: $(y^*(p))$.
4. If p stays constant, how does y^* respond to an increase in H ?
5. What is the profit function of this firm?