Econ 101A
Section 5 & 6

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1 Envelope Theorem

1.1 The Theorem

Suppose $x^*(p)$ solves the following optimization problem:

$$\max_x f(x;p)$$

In this example:

- p is **exogenous**, meaning that it is determined independently of other variables.
- $x^*(p)$ is **endogenous**, as it is a function of other (exogenous) variables p.

Define $F(p) = f(x^*(p), p)$ (this expression is now just a function of p, since at the optimum, x is endogenously determined by p). Then, we have:

$$\begin{aligned}
\frac{dF(p)}{dp} &= \underbrace{\left.\frac{\partial f(x;p)}{\partial x}\right|_{x=x^*}}_{\text{By FOC, this is 0}} \times \frac{\partial x^*}{\partial p} + \left.\frac{\partial f(x;p)}{\partial p}\right|_{x=x^*} \\
&= \left.\frac{\partial f(x;p)}{\partial p}\right|_{x=x^*}
\end{aligned}$$

1.2 What does this mean?

It's very easy to calculate the derivatives at the maximum of an objective function! The total differential of the function $F(p)$ with respect to p equals the partial derivative with respect to p at $x = x^*$ (this means that you can disregard the indirect effects!).

- When differentiating a function with respect to an exogenous variable, typically you do the following:
  1. Write the function in terms of your exogenous variable only (in the above notation, this means that you would substitute x with it's expression in terms of p, so $x^*(p)$).
  2. Then, take the derivative with respect to the exogenous variable (i.e. p)
- Using the Envelope Theorem, you can calculate the derivative at the optimum as follows:

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1. Differentiate the objective function with respect to the desired exogenous variable (holding all other variables constant, including the endogenous variable that maximizes your objective)
2. Now you have an expression (the derivative) written in terms of your endogenous variables. Plug in their optimal values of the endogenous variables (i.e. $x^*(p)$), and you're done!

1.3 Exercise

A student wants to maximize her grade (g) on an essay. She selects her time spent writing (t) in order to achieve her top grade. Her time spent writing is a function of the noise at her local library (q), which is out of her control (or *exogenous*):

$$g = -(2t - q)^2 - t^2 + 100$$

In short, she is solving the following problem: $\max_t g(t; q)$. The F.O.C. for this problem is given by: $-4(2t - q) - 2t = 0$ (check this). From our FOCs, we can solve for our optimal t in terms of q, finding that $t^*(q) = \frac{2q}{5}$. Using this information, calculate the derivative of g with respect to q at the optimal level of t (i.e., the rate at which her grade changes in response to a change in library "quality") in two different ways:

1. Plug in $t^*(q) = \frac{2q}{5}$ for t into our original expression for g, and then differentiate with respect to q.
2. (Envelope Theorem) Calculate the partial derivative $\frac{\partial g}{\partial q}$, and evaluate when $t = t^*(q)$.
3. How do these derivatives compare?

2 Constrained Maximization with Comparative Statics

Suppose that you want to maximize $f(x; p)$ such that $h(x; p) = 0$, and x is an n-dimensional vector. [1] The steps:

1. Write the Lagrangian function:

$$\mathcal{L}(\lambda, x; p) = f(x; p) - \lambda h(x; p).$$

   Note that λ is joining your x's as an additional endogenous variable.

2. Your (necessary) first order conditions (setting $\bigtriangledown \mathcal{L}(\lambda^*, x^*; p)$ will now give your $n + 1$ equations in $n + 1$ unknowns. To get your candidate solutions, take the first derivatives of $\mathcal{L}$ with respect to all x_i's and λ and set all $n + 1$ of these equations equal to zero. You will notice that you will get the following:

$$h(x^*; p) = 0$$

$$\frac{\partial f(x^*; p)}{\partial x_i} - \lambda^* \times \frac{\partial h(x^*; p)}{\partial x_i} = 0 \qquad \textit{for all } i$$

3. To check the sufficient second order condition for a constrained maximum (or minimum), we use the bordered Hessian, which is calculated simply as the Hessian of $\mathcal{L}(\lambda, x; p)$. The determinant of the bordered Hessian is positive for a constrained maximum.

4. Now we want to calculate $dx_i^*(p)/dp$. We need to apply multivariate version of Implicit Function Theorem because we have more than 1 endogenous variables (for a general version, see Appendix).

[1] Here, we will assume that h is scalar-valued, meaning that we only have one constraint. In lecture, we were more general, and allowed ourselves to have more than one constraint.

The intuition behind IFT still applies. First, find a set of **equations** (FOC):

$$\underbrace{\mathbf{H}(x,p)}_{(n+1)\times 1} = \begin{cases} h(x;p) \\ \partial f(x;p)/\partial x_1 - \lambda \cdot \partial h(x;p)/\partial x_1 \\ \cdots \\ \partial f(x;p)/\partial x_n - \lambda \cdot \partial h(x;p)/\partial x_n \end{cases} = \underbrace{\mathbf{0}}_{(n+1)\times 1}$$

Then, if you change the value p, it would have a direct effect $D_p\mathbf{H}(x,p)$, and an indirect effect through all x_i (p can first affects any x_i, which we have $n+1$ endogenous variables, and then each of the $n+1$ endogenous variables can affect each of the $n+1$ equations in $\mathbf{H}$). Two effects cancel out eventually because we're operating on a set of equations:

$$\underbrace{D_x\mathbf{H}(x,p)}_{(n+1)\times(n+1)} \underbrace{D_p x}_{(n+1)\times 1} = -\underbrace{D_p\mathbf{H}(x,p)}_{(n+1)\times 1}$$

Now this is a classic system of equations we see in linear algebra. We can use Cramer's Rule (see Appendix) to solve for $dx_i^*(p)/dp$.

5. Now we want to calculate $df(x^*(p);p)/dp$. Envelope Theorem applies to the partial derivative of the *Lagrangian*, NOT the partial in the objective function. In other words, the Envelope Theorem for constrained optimization is as follows:

$$\frac{df(x^*(p);p)}{dp} = \frac{\partial \mathcal{L}(\lambda^*(p), x^*(p);p)}{\partial p}$$

2.1 Exercise

Suppose you are taking only two classes: literature and economics. It is finals week and you know you can study productively for T hours a day. You must choose how to split these hours between studying for literature (which involves a lot of reading) and studying economics. Your goal is to maximize your GPA. An exogenous variable, the amount of free coffee you get, helps you read more efficiently– but it does not affect your studying of economics. Assume that you know exactly how your daily studying of each subject will determine your GPA this semester:

$$GPA = \frac{2}{3}\sqrt{C}\sqrt{h_L} + \frac{2}{3}\sqrt{h_E}$$

where h_L = hours studying literature, h_E = hours studying economics, and C = cups of coffee.

1. What would be the Lagrangian for this maximization problem? Find the potential optimum and check the second order condition.

2. How will the hours spent studying literature change if the exogenous amount of coffee increases? Use the IFT to answer this question (you'll either need to use Cramer's rule or to inverse a 3 by 3 matrix). Can you solve for the explicit form of h_L^* as a function of C to verify this answer?

3. How will your maximized GPA change if the exogenous amount of coffee C increases? What if the number of total productive hours T increases? Use Envelop Theorem (the constrained version) to answer this question.

3 Indirect Utility Function

- **Definition:**

  The indirect utility function takes prices and income as its arguments and gives you the maximum achievable utility. Here is the two-dimensional case (2 prices, 2 goods):

$$v(\boldsymbol{p}, M) = u(\boldsymbol{x_1^*}(\boldsymbol{p}, M), \boldsymbol{x_2^*}(\boldsymbol{p}, M); \boldsymbol{p}, M)$$

- **Properties:**

1. As we increase M the set of affordable bundles (the budget set) increases. Maximizing a function over a larger set will only increase the value at the optimum. Thus, $\partial v(\mathbf{p}, M)/\partial M \geq 0$.

2. As we increase p_i, the set of affordable bundles (the budget set) decreases. Maximizing a function over a smaller set will only decrease the value at the optimum. Thus, $\partial v(\mathbf{p}, M)/\partial p_i \leq 0$.

3. Note that we can use the Envelope Theorem to find how our (indirect) utility changes with prices and our income. For example:

$$\frac{\partial v(\boldsymbol{p}, M)}{\partial p_1} = \frac{\partial \mathcal{L}(\lambda^*, x_1^*, x_2^*; p_1, p_2, M)}{\partial p_1} = \frac{\partial u(x_1^*, x_2^*)}{\partial p_1} - \lambda^* x_1^*$$

Our prices usually don't appear in the utility function, in which case, the above expression reduces to the following: $\frac{\partial v(\boldsymbol{p},M)}{\partial p_1} = -\lambda^* x_1^*$.
Likewise, $\frac{\partial v(\boldsymbol{p},M)}{\partial M} = \lambda^*$. Thus we can interpret λ as marginal utility of an extra dollar of consumption expenditure, or marginal utility of "income".

Instead of maximizing utility given expenditure M, we can equivalently ask: what is the *minimum* expenditure needed to reach a target utility level $\bar{u}$? This leads to the expenditure function $e(p, \bar{u})$ and Hicksian (compensated) demand.

4 Expenditure Minimization

4.1 The dual problem

Instead of maximizing utility subject to a budget, we can *minimize expenditure* subject to achieving a target utility level $\bar{u}$:

$$e(p, \bar{u}) = \min_{x \in \mathbb{R}_+^L} p \cdot x \quad \text{s.t. } u(x) \geq \bar{u}.$$

Any optimizer is called **Hicksian (compensated) demand**:

$$h(p, \bar{u}) \in \arg \min_{u(x) \geq \bar{u}} p \cdot x.$$

4.2 Lagrangian and key envelope result

$$\mathcal{L}(x, \mu; p, \bar{u}) = p \cdot x + \mu(\bar{u} - u(x)), \qquad \mu \geq 0.$$

By the Envelope Theorem (applied to the *Lagrangian*):

$$\frac{\partial e(p, \bar{u})}{\partial p_i} = h_i(p, \bar{u}) \qquad \text{(Shephard's lemma).}$$

Also,

$$\frac{\partial e(p, \bar{u})}{\partial \bar{u}} = \mu^*(p, \bar{u}) \geq 0,$$

so μ^* is the marginal cost (in dollars) of raising the required utility.

4.3 Duality link

$$v(p, M) = \max\{\bar{u} : \ e(p, \bar{u}) \leq M\}, \qquad x(p, M) = h\big(p, v(p, M)\big).$$

4.4 Exercise

Let $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ with $\alpha \in (0, 1)$ and prices $p = (p_1, p_2) \gg 0$.

1. Solve the expenditure minimization problem and find Hicksian demands $h_1(p, \bar{u})$ and $h_2(p, \bar{u})$.
2. Compute $e(p, \bar{u}) = p \cdot h(p, \bar{u})$.
3. Verify Shephard's lemma by checking that $\partial e / \partial p_i = h_i$ for $i = 1, 2$.