Econ 101A
Section 10

Clotaire Boyer

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Consumer preferences and utility

The first step in microeconomic analysis is to model how consumers rank alternative bundles of goods. A preference relation $\succeq$ on a commodity set X is interpreted as "$x \succeq y$ means that bundle x is at least as good as bundle y". Several properties:

- **Rationality.** A preference relation is *rational* if it is complete and transitive. Completeness means that for any two bundles $x, y \in X$ we have either $x \succeq y$ or $y \succeq x$ (or both), while transitivity requires that $x \succeq y$ and $y \succeq z$ implies $x \succeq z$. Rationality ensures that choices are coherent.
- **Monotonicity.** A preference relation is monotonic if more is never worse: whenever $x \geq y$ (coordinate-wise), we have $x \succeq y$. If goods are "goods" rather than "bads", monotonicity is a natural assumption.
- **Convexity.** Preferences are convex if consumers prefer averages to extremes: whenever $x \succeq z$ and $y \succeq z$, then $tx+(1-t)y \succeq z$ for all $t \in [0,1]$. Convexity implies diminishing marginal rate of substitution (MRS).
- **Continuity.** If $x \succ y$, then small perturbations of x and y preserve the strict preference. Continuity rules out abrupt jumps in preferences.

Under these assumptions, preferences can be represented by a *utility function* $u\colon X \to \mathbb{R}$. A function u represents $\succeq$ if $x \succeq y$ if and only if $u(x) \geq u(y)$. The representation theorem states that any rational, continuous preference admits a continuous utility representation. Utility functions are ordinal: any strictly increasing transformation $f(u)$ represents the same preferences.

Examples. Common functional forms capture different shapes of indifference curves:

- *Perfect substitutes*: $u(x_1, x_2) = \alpha x_1 + \beta x_2$. Indifference curves are straight lines with slope $-\alpha/\beta$ and MRS is constant
- *Perfect complements*: $u(x_1, x_2) = \min\{x_1, x_2\}$. Consumers only care about the minimum of the two goods; the optimal bundle satisfies $x_1 = x_2$.
- *Cobb–Douglas*: $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$ with $\alpha, \beta > 0$. Indifference curves are convex and MRS diminishes; optimal demands have constant expenditure shares.

The *marginal rate of substitution* (MRS) measures the slope of an indifference curve: for a twice differentiable utility function u,

$$\mathrm{MRS}_{1,2}(x_1, x_2) \equiv -\frac{\mathrm{d}x_2}{\mathrm{d}x_1}\bigg|_{u(x)=\overline{u}} = \frac{u_{x_1}}{u_{x_2}},$$

where u_{x_i} denotes the marginal utility with respect to good i. Diminishing MRS is equivalent to convex preferences.

Implicit Function Theorem

Many comparative-statics questions require finding the derivative of an implicitly defined function. Suppose $g(x, y) = 0$ defines y as a function of x. If g is continuously differentiable and $g_y(x_0, y_0) \neq 0$ at a point (x_0, y_0), the *implicit function theorem* ensures that in a neighbourhood of (x_0, y_0) one can write $y = \phi(x)$ with

$$\phi'(x) = -\frac{g_x}{g_y}\bigg|_{(x,\phi(x))}.$$

In consumer theory, indifference curves are defined by $u(x_1, x_2) = \bar{u}$. Applying the IFT gives the slope of the indifference curve:

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1}\bigg|_{u(x)=\bar{u}} = -\frac{u_{x_1}}{u_{x_2}},$$

which is the negative of the marginal rate of substitution.

Envelope theorem for unconstrained optimisation

For an unconstrained maximisation problem $\max_x f(x, p)$ with smooth f and unique optimizer $x^*(p)$, first-order conditions imply $\nabla_x f(x^*(p), p) = 0$. Defining $F(p) = f(x^*(p), p)$, the envelope theorem states that

$$\frac{\mathrm{d}F}{\mathrm{d}p} = \frac{\partial f}{\partial p}\bigg|_{(x^*(p),p)},$$

because the indirect effect through $x^*(p)$ vanishes at the optimum. This is a special case of the constrained envelope theorem. It allows one to compute how the maximised value changes with a parameter without differentiating the optimal choice.

Second-order conditions for unconstrained problems

First-order conditions identify stationary points where $\nabla_x f(x) = 0$. To determine whether such a point is a local maximum or minimum, examine the Hessian matrix $H_f(x)$ of second partial derivatives. In a single variable, $f''(x^*) < 0$ implies a strict local maximum and $f''(x^*) > 0$ implies a strict local minimum. In higher dimensions, x^* is a strict local maximum if $H_f(x^*)$ is negative definite (all eigenvalues negative), and a strict local minimum if $H_f(x^*)$ is positive definite. For constrained problems, the bordered Hessian criterion extends these ideas; but in the unconstrained case the sign of the Hessian suffices. In this class you do not need to work with matrices because we would stay in a two goods space and give you the SOCs in linear form:

$$u_{x_1x_1}(x^*) < 0 \quad \text{and} \quad u_{x_1x_1}(x^*)\, u_{x_2x_2}(x^*) - \big(u_{x_1x_2}(x^*)\big)^2 > 0$$

Utility maximisation and the Lagrangian method

Consumers choose the most preferred bundle they can afford. Given strictly positive prices $p = (p_1, p_2)$ and income M, the budget set is $\{(x_1, x_2) \mid p_1x_1 + p_2x_2 \leq M\}$. Under monotonic preferences, the budget constraint binds, so the problem is

$$\max_{x_1,x_2} u(x_1, x_2) \quad \text{s.t.} \quad p_1x_1 + p_2x_2 = M.$$

To handle the equality constraint we write the Lagrangian

$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) - \lambda\,(p_1x_1 + p_2x_2 - M),$$

and the first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_i} = u_{x_i} - \lambda p_i = 0 \quad (i = 1, 2), \tag{1}$$

$$p_1x_1 + p_2x_2 = M. \tag{2}$$

Setting the two marginal-utility conditions equal implies $u_{x_1}/u_{x_2} = p_1/p_2$. Hence the MRS along the optimal bundle equals the ratio of prices: the budget line is tangent to the highest attainable indifference curve. Solving the system yields the Marshallian demand $x^*(p, M)$, e.g. for Cobb–Douglas utility $u(x_1, x_2) = x_1^\alpha x_2^\beta$ one obtains $x_1^* = \frac{\alpha M}{(\alpha+\beta)p_1}$ and $x_2^* = \frac{\beta M}{(\alpha+\beta)p_2}$.

Lagrange multiplier theorem. For a general maximisation problem

$$\max_{x\in\mathbb{R}^n} f(x) \quad \text{s.t.} \quad h(x) = 0,$$

if $\nabla h(x^*) \neq 0$ at the solution x^*, then there exists a multiplier λ such that x^* maximises the Lagrangian $\mathcal{L}(x, \lambda) = f(x) - \lambda h(x)$. The first-order conditions are $\partial\mathcal{L}/\partial x_i = 0$ and $h(x) = 0$. Geometrically, $\nabla f(x^*)$ is parallel to $\nabla h(x^*)$. The multiplier λ measures the shadow value of relaxing the constraint.

Marshallian demand and indirect utility. Define the indirect utility function by $v(p, M) = u\big(x^*(p, M)\big)$ the maximum attainable utility at prices p and income M. Marshallian demand $x^*(p, M)$ is homogeneous of degree zero in (p, M) and satisfies important comparative-statics properties. An increase in income (with prices fixed) typically increases demand for normal goods, while an increase in the price of a good usually decreases its demand. Goods i and j are gross complements if $\partial x_i^*/\partial p_j < 0$ and gross substitutes if $\partial x_i^*/\partial p_j > 0$.

Not all goods are normal Normal goods products that experience increased demand as consumer income rises and decreased demand when income fallbut not all goods behave this way. A Giffen good is a rare case where a price increase leads to higher quantity demanded because the negative income effect outweighs the substitution effect. Giffen goods for example are inferior goods (products for which demand decreases when consumer income rises) with strongly positive income shares. They are often mixed up with Veblen goods (e.g. luxury watches) which are goods for which higher prices make the product more desirable by signalling status; their demand curves slope upward not because of income effects but because the price itself enters preferences. While both exhibit upward-sloping demand, Giffen goods arise within the standard framework, whereas Veblen goods violate the assumption that utility does not depend on prices.

Expenditure minimisation and Hicksian demand The dual of the utility–maximisation problem is the *expenditure minimisation* problem: given prices p and a target utility level u_0, choose a bundle x to minimise cost subject to $u(x) \geq u_0$. The solution $h(p, u_0)$ is the *Hicksian (compensated) demand*, and the minimum expenditure required to achieve u_0 is the *expenditure function* $e(p, u_0) = p \cdot h(p, u_0)$. Hicksian demand depends only on preferences and prices, not income. Duality implies $x^*(p, e(p, u_0)) = h(p, u_0)$. By Shephard's lemma, the derivative of the expenditure function with respect to price equals Hicksian demand: $\partial e(p, u_0)/\partial p_i = h_i(p, u_0)$.

Envelope theorem and comparative statics

Suppose we have a parameterised maximisation problem

$$\max_{x} f(x; p) \quad \text{s.t.} \quad h(x; p) = 0,$$

and let $x^*(p)$ denote the optimising choice. Define $F(p) = f\big(x^*(p); p\big)$, the optimised objective. Differentiating F with respect to the parameter p is non-trivial because x^* depends on p. The envelope theorem states that one can ignore the indirect effect and differentiate the Lagrangian directly:

$$\frac{\mathrm{d}F}{\mathrm{d}p} = \frac{\mathrm{d}\mathcal{L}\big(x^*(p); p\big)}{\mathrm{d}p} = \frac{\partial f(x; p)}{\partial p} - \lambda\,\frac{\partial h(x; p)}{\partial p},$$

where λ is the Lagrange multiplier at the optimum. The theorem allows comparative statics on optimised values without differentiating the optimal choice. For example, the marginal effect of income on indirect utility is $\partial v(p, M)/\partial M = \lambda$, and because $u'_{x_i}/p_i = \lambda$ the multiplier is positive.

Intertemporal choice

As an application, consider a two-period consumption–saving problem. An individual with income (M_0, M_1) chooses consumption (c_0, c_1) to maximise discounted utility $U(c_0) + \frac{1}{1+\delta}U(c_1)$ subject to the intertemporal budget constraint $c_0 + \frac{1}{1+r}c_1 \leq M_0 + \frac{1}{1+r}M_1$. Since utility is strictly increasing, the budget binds, and the Lagrangian is

$$\mathcal{L}(c_0, c_1, \lambda) = U(c_0) + \frac{1}{1+\delta}U(c_1) - \lambda\Big(c_0 + \tfrac{c_1}{1+r} - M_0 - \tfrac{M_1}{1+r}\Big).$$

The first-order conditions equate the marginal utilities across periods:

$$\frac{U'(c_0)}{U'(c_1)} = \frac{1+r}{1+\delta},$$

and combined with the budget constraint this yields the optimal consumption path. When the interest rate equals the rate of time preference ($r = \delta$), optimal consumption is smoothed across periods; solving gives $c_0^* = c_1^* = (1+r)\big[M_0 + \frac{M_1}{1+r}\big]/(2+r)$. Comparative statics via the implicit function theorem shows that $\partial c_0^*(r, M)/\partial M_0 > 0$, so consumption in period 0 is a normal good. Can you solve the full model? What if we had 3 periods? We would not test you on N.
Do you remember how this problem will differ if you add some present bias factor β? Try to solve the problem from the perspective of period 0 vs 1 in the classic problem above and in the inconsistent time preferences cases to make the difference very clear.

Problems

Problem 1: Preferences and demand

Problem 1 *Consider a consumer with utility function $u(x, y) = \sqrt{x} + c\sqrt{y}$ defined on $x, y \geq 0$. The price of good x is $p_x > 0$, the price of y is normalised to 1, and income is $M > 0$.*

(a) For which values of c is u monotonic in each good? Explain using the definition of monotonicity.

(b) Derive the MRS between x and y and interpret its economic meaning. Confirm that the MRS does not depend on the representation of preferences.

(c) Write the consumer's maximisation problem and construct the Lagrangian. Derive the first-order conditions and solve for the Marshallian demand (x^*, y^*) as functions of p_x, M and c.

(d) Determine whether goods x and y are gross complements or gross substitutes by analysing the sign of $\partial x^*/\partial p_y$. Is x a normal or inferior good? Support your answer with comparative statics using the envelope theorem.

Problem 2: Intertemporal choice and the envelope theorem

Problem 2 *An agent lives for two periods and has utility $U(c_0) + \frac{1}{1+\delta}U(c_1)$ with $U' > 0$ and $U'' < 0$. The intertemporal budget constraint is $c_0 + \frac{c_1}{1+r} = M_0 + \frac{M_1}{1+r}$. Let $c_0^*(r, M_0, M_1)$ denote the optimal period-0 consumption.*

(a) Derive the first-order conditions using the Lagrangian method and show that $U'(c_0)/U'(c_1) = (1 + r)/(1 + \delta)$.

(b) Use the envelope theorem to compute $\frac{\partial V}{\partial M_0}$, where $V(r, M_0, M_1) = U\big(c_0^*(r, M_0, M_1)\big) + \frac{1}{1+\delta}U\big(c_1^*(r, M_0, M_1)\big)$ is the indirect utility. Interpret the shadow price of the budget constraint.

(c) Suppose $U(c) = \ln c$, $r = \delta$ and $M_0 = 1, M_1 = 2$. Solve explicitly for (c_0^*, c_1^*) and check whether consumption is smoothed. Verify that $\partial c_0^*/\partial M_0 > 0$.