Micro Econ Theory : Week 2

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Unconstrained Optimization

1. Univariate Review and Concavity

A stationary point x^* satisfies $f'(x^*) = 0$. To confirm it is a local maximum, we check the SOC: $f''(x^*) < 0$. If $f''(x) \leq 0$ for all x, the function is globally concave, and the FOC identifies the global maximum.

Exercise 1.1: Consider $\max_x f(x;p) = (p-x)(x-1)$ where p is a parameter.

1. Find x^* and verify the SOC.
2. Is the function concave? Is x^* a global maximum?

2. Multivariate Basics

For $f(x_1, x_2)$, the FOC requires $\nabla f = 0$ (both $f_{x_1} = 0$ and $f_{x_2} = 0$). The SOC for a maximum requires $f_{x_1,x_1} < 0$ and the determinant $D = f_{11}f_{22} - (f_{12})^2 > 0$.

Exercise 1.2: Maximize $f(x, y; a, b) = ax^2 - x + by^2 - y$ where $a, b \neq 0$.

1. Solve for (x^*, y^*).
2. Provide sufficient conditions on a and b for this point to be a maximum.
3. Under what conditions is f globally concave?

Exercise 1.3: Saddle Points
Consider the function $f(x, y; a, b, c) = -(x-a)^2 - (y-b)^2 + cxy$, where c is a parameter representing the interaction between x and y.

1. Find the stationary point (x^*, y^*) in terms of a, b, and c.
2. Use the determinant condition $f_{xx}f_{yy} - (f_{xy})^2 > 0$ to find the range of values for c that ensures (x^*, y^*) is a local maximum.
3. *Intuition:* If $|c| \geq 2$, the interaction term is so strong that the surface "twists" into a saddle point, meaning no local maximum exists.

IFT and Constraints

3. Implicit Function Theorem (IFT)

When we cannot solve $f(x;p) = 0$ explicitly for x, we use IFT to find how x responds to p:

$$\frac{dx}{dp} = -\frac{f_p}{f_x} \text{ (provided } f_x \neq 0)$$

Exercise 2.1: Let $f(x;p) = 1 - xp - p \cdot \exp(x) = 0$. Find $\frac{dx}{dp}$ at $(0,1)$.

Exercise 2.2: Use IFT to prove that for a utility function $u(x_1, x_2) = \bar{u}$, the slope of the indifference curve (MRS) is $\frac{u_{x_1}}{u_{x_2}}$.

4. Constrained Optimization (Lagrangians)

To solve $\max f(x)$ s.t. $h(x) = 0$, we build the Lagrangian $\mathcal{L} = f(x) - \lambda h(x)$. The FOCs are $\frac{\partial \mathcal{L}}{\partial x_i} = 0$ and $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$.

Exercise 2.3: Maximize $U(l,c) = \phi(l) + c$ subject to $p \cdot c = M + w(24 - l)$

1. Write the budget constraint and the Lagrangian.
2. Solve the FOCs and show that $\phi'(l^*) = w/p$.
3. If w increases, what happens to optimal leisure l^*? (Use IFT on the FOC)

Exercise 2.4: Comparative Statics via IFT on FOCs
Return to Elisabetta's labor supply problem where the optimality condition is $F(l;w) = \phi'(l) - w/p = 0$.

1. Assume $\phi''(l) < 0$ (diminishing marginal utility of leisure).
2. Treat the FOC as an implicit function. Use the IFT formula $\frac{\partial l^*}{\partial w} = -\frac{F_w}{F_l}$ to find the sign of the effect of a wage increase on leisure.
3. *Economic Intuition:* Does Elisabetta work more or less when her wage increases? Explain why the sign of $\phi''(l)$ is crucial here.